POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-10B(i) : ADVANCED FUNCTIONAL ANALYSIS

(Pure Mathematics)

(Spl. Paper)

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

 a) Give an example with proper justification of a balanced set in a vector space which is not absorbing.

 b) If M is a subset of a topological vector space X, then prove that $\alpha \overline{M} = \overline{\alpha M}$ for any scalar α.

 c) Suppose A is a convex absorbing set containing $\underset{\sim}{0}$ of a vector space X and let p_A be the Minkowski functional for A. Then prove that $p_A(\lambda x) = \lambda\, p_A(x)$ for all scales $\lambda \geq 0$ and for all $x \in X$.

 d) In a normed linear space, show that weak convergence does not imply strong convergence.

 e) Let X be a normed linear space, $z \in X$ and $f \in X^*$. Show that $T : X \to X$ defined by $T(x) = f(x)z,\ x \in X$ is a compact linear operator.

 f) Show that the set of all invertible elements of a Banach algebra X with identity e forms a group under multiplication.

 g) Let H be complex Hilbert space and $T \in B(H)$. Then prove that T can be expressed uniquely as $T = A + iB$ where A, B are self adjoint operators on H.

2. a) Prove that a topological vector space has a balanced local base. 2

   b) Prove that the following statements are equivalent in a topological vector space X :

   (i) A subset B of X is bounded.

   (ii) If $\{x_n\}$ is any sequence in B and $\{\alpha_n\}$ is any sequence of scalars with $\lim_{n \to \infty} \alpha_n = 0$, then $\{\alpha_n x_n\}$ converges to $\underset{\sim}{0}$ in X.

   4

   c) Let X be a topological vector space. Prove that

   (i) if C is a convex subset of X, so are Int C and $\overline{C}$.

   (ii) if B is a balanced subset of X, so is $\overline{B}$. If also $\underset{\sim}{0} \in \text{Int}\, B$, then Int B is balanced. 2 + 2

3. a) If K and C are subsets of a topological vector space X, K is compact, C is closed and $K \cap C = \Phi$, then prove that there exists a neighbourhood V of $\underset{\sim}{0}$ in X such that $(K+V) \cap (C+V) = \Phi$. 5

   b) Prove that every locally compact topological vector space is finite dimensional. 5

4. a) Let f be a non-zero linear functional on a topological vector space X. Prove that the following statements are equivalent :

   (i) f is continuous.

   (ii) $\ker(f) = \{x \in X : f(x) = 0\}$ is closed.

   (iii) $\ker(f)$ is not dense in X.

   (iv) f is bounded in some neighbourhood $\underset{\sim}{0}$ in X. 6

   b) Let $\mathscr{B}$ be a convex balanced local base in a topological vector space X. Associate to every $V \in \mathscr{B}$ its Minkowski functional p_v.

   Then prove that

   (i) $V = \{x \in X : p_v(x) < 1\}$ for very $V \in \mathscr{B}$

   (ii) $\{p_v : V \in \mathscr{B}\}$ is a separating family of continuous seminorms on X. 4

5. a) Prove that the conjugate space of l_p is isomorphic to the sequence space l_q where $1 < p,\ q < \infty$ and $p^{-1} + q^{-1} = 1$. 4

b) Show that in a normal linear space X, the set M of all best approximations to a given $x \in X$ out of a subspace Y is a convex set. 3

c) Let X be a normed linear space with a strictly convex norm and G be a subspace of X. If $x \in X$, then prove that there is at most one best approximation to x out of the elements of G. 3

6. a) Let X and Y be two normed linear spaces. When is a transformation $T : X \to Y$ called compact ? If $T, S : X \to Y$ are compact operators and α is a scalar then prove that $T + S$, αT are compact. 1 + 3

b) Show that the spectrum $\sigma(T)$ of a bounded self adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies in the closed interval [m, M] on the real axis where $m = \inf_{\|x\|=1} < Tx, x >$, $M = \sup_{\|x\|=1} < Tx, x >$. 3

c) Let H be a complex Hilbert space and let P_1, P_2 be orthogonal projections on the closed subspaces M_1 and M_2 respectively. Show that $P_1 P_2$ is an orthogonal projection if and only if $P_1 P_2 = P_2 P_1$. 3

7. a) Let X be a Banach algebra with identity e. If $x \in X$ satisfies $\|x\| < 1$ then prove that $(e - x)$ is invertible and $(e - x)^{-1} = e + \sum_{j=1}^{\infty} x^j$. 4

b) State and prove Gelfand-Mazur theorem. 3

c) Prove that a subspace M of a normed linear space X is weekly closed if and only if it is strongly closed. 3